On the Number of Congruent Simplices in a Point Set*

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ABSTRACT

We derive improved bounds on the number of k-dimensional simplices spanned by a set of n points in $\mathbb{R}^d$ that are congruent to a given k-simplex, for $k \le d-1$. Let $f_k^{(d)}(n)$ be the maximum number of k-simplices spanned by a set of n points in $\mathbb{R}^d$ that are congruent to a given k-simplex. We prove that $f_2^{(3)}(n) = O(n^{5/3} \cdot 2^{O(\alpha^2(n))})$, $f_2^{(4)}(n) = O(n^{2+\varepsilon})$, $f_2^{(5)}(n) = \Theta(n^{7/3})$, and $f_3^{(4)}(n) = O(n^{9/4+\varepsilon})$. We also derive a recurrence to bound $f_k^{(d)}(n)$ for arbitrary values of k and d, and use it to derive the bound $f_k^{(d)}(n) = O(n^{d/2})$ for $d \le 7$ and $k \le d-2$. Following Erdős and Purdy, we conjecture that this bound holds for larger values of d as well, and for $k \le d-2$.

1. INTRODUCTION

Let P be a set of n points in $\mathbb{R}^d$, and let Δ_0 be a prescribed k-dimensional simplex, for some $1 \le k \le d-1$. Let $f_k^{(d)}(P, \Delta_0)$ be the number of k-simplices spanned by P that are congruent to Δ_0. Set

$$f_k^{(d)}(n) = \max f_k^{(d)}(P, \Delta_0),$$

where the maximum is taken over all sets of n points in $\mathbb{R}^d$ and over all k-simplices in $\mathbb{R}^d$. We wish to obtain sharp bounds for $f_k^{(d)}(n)$.

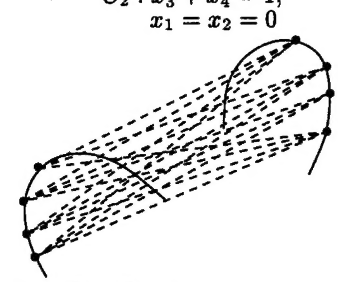

Figure 1: A construction for $f_1^{(4)}(n) = \Omega(n^2)$.

The case $k = 1$ is the well-studied problem of *repeated distances*, originally considered by Erdős [7] in 1946: How many pairs of points of P lie at a prescribed distance from each other. This special case is interesting only for $d = 2, 3$ because $f_1^{(d)}(n) = \Theta(n^2)$ for $d \ge 4$. Indeed, as observed by Lenz [11], one can construct in $\mathbb{R}^4$ two orthogonal unit circles $C_1 : x_1^2 + x_2^2 = 1, x_3 = x_4 = 0$ and $C_2 : x_1 = x_2 = 0, x_3^2 + x_4^2 = 1$ and place $n/2$ points on each of the two circles. The distance between any two points $p \in C_1$ and $q \in C_2$ is $\sqrt{2}$, thereby obtaining a set P of n points with $\Omega(n^2)$ pairs of points at distance $\sqrt{2}$. The known upper bounds for $d = 2, 3$ are $f_1^{(2)}(n) = O(n^{4/3})$ [6, 15, 16] and $f_1^{(3)}(n) = O(n^{3/2}\beta(n))$ [6], where $\beta(n) = 2^{\Theta(\alpha^2(n))}$ is a slowly growing function of n, defined in terms of the inverse Ackermann's function $\alpha(n)$. However, neither of these bounds is known to be tight. The best known lower bounds are $f_1^{(2)}(n) = n^{1+\Omega\left(\frac{1}{\log\log n}\right)}$ and $f_1^{(3)}(n) = \Omega(n^{4/3} \log\log n)$; see e.g. [12].

Note that we have excluded the cases $k = 0$ and $k = d$. The case $k = 0$ is uninteresting because, trivially, $f_0^{(d)}(n) = n$. The case $k = d$ is also uninteresting because one easily has $f_d^{(d)}(n) = O(f_{d-1}^{(d)}(n))$. It is conceivable, though, that $f_d^{(d)}(n)$ is significantly smaller than $f_{d-1}^{(d)}(n)$. However, we are not aware of any instance where this has been shown to be the case. Another easy observation is that $f_k^{(d)}(n) = \Theta(n^{k+1})$ for any $k \le \lfloor d/2 \rfloor - 1$. The upper bound is trivial, and the lower bound can be proved by generalizing the construction for the case $k = 1$, namely, by placing

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the points of P on $k+1$ mutually orthogonal unit-radius circles centered at the origin. Erdős and Purdy [9] proved that $f_2^{(3)}(n) = O(n^{19/9})$. The bound was later improved by Akutsu *et al.* [2] to $O(n^{9/5})$ and then by Brass [5] to $O(n^{7/4})$. Akutsu *et al.* [2] also proved that $f_2^{(4)}(n) = O(n^{65/23+\varepsilon})$ and $f_3^{(4)}(n) = O(n^{66/23+\varepsilon})$ for any $\varepsilon > 0$.[1] Erdős and Purdy [10] conjectured that $f_k^{(d)}(n) = O(n^{d/2})$ for even values of d.

We prove that $f_2^{(3)}(n) = O(n^{5/3}\beta^{4/3}(n))$, $f_2^{(4)}(n) = O(n^{2+\varepsilon})$, $f_2^{(5)}(n) = \Theta(n^{7/3})$, and $f_3^{(4)}(n) = O(n^{9/4+\varepsilon})$. The best lower bound that we know for $f_2^{(3)}(n)$ is $\Omega(n^{4/3})$. This is obtained by placing one point at the origin and $n-1$ additional points on the unit sphere, so that there are $\Omega(n^{4/3})$ pairs of those $n-1$ points at distance $\sqrt{2}$ from each other (see [8] for such a construction). The bound on $f_2^{(4)}(n)$ is almost tight because it can be shown that $f_2^{(4)}(n) = \Omega(n^2)$ (e.g., add the origin to the set of points in Lenz' construction).

We also derive a recurrence for $f_k^{(d)}(n)$ for general values of k and d. The solution of this recurrence is $O(n^{\zeta(d,k)+\varepsilon})$, where $\zeta(d,k)$ is a rather complicated function of d and k. Although we are currently unable to provide sharp explicit bounds for $\zeta(d,k)$, for arbitrary values of k and d, we can prove that $\zeta(d,k) \le d/2$ for $d \le 7$ and $k \le d-2$. We conjecture that $\zeta(d,k) \le d/2$ for all d and $k \le d-2$. Proving this bound on $\zeta(d,k)$ will (almost) settle in the affirmative the above-mentioned conjecture of Erdős and Purdy.

A novel feature of our analysis is a round-robin recurrence scheme. In each round of this scheme some of the given points are treated as points while others are treated as spheres of various radii (equal to the lengths of the corresponding edges of the given simplex Δ). The recurrence then follows from a space partitioning process, based on a $(1/r)$-cutting of these sets of spheres; see Sections 3 and 5 for details.

The problem is motivated by the problem of *exact pattern matching*: We are given a set E of n points in $\mathbb{R}^d$ and a "pattern set" P of $m \le n$ points (in most applications m is much smaller than n), and we wish to determine whether E contains a congruent copy of P, or, alternatively, to enumerate all such copies. A commonly used approach to this problem is to take a simplex Δ_0 spanned by some points of P, and find all congruent copies of Δ_0 that are spanned by E. For each such copy Δ, take the Euclidean motion(s) that map Δ_0 to Δ, and check whether all the other points of P map to points of E under that motion. The efficiency of such an algorithm depends on the number of congruent copies of Δ_0 in E. Using this approach, de Rezende and Lee [13] developed an $O(mn^d)$ algorithm to determine whether E contains a congruent copy of P. For $d=3$, Brass recently developed an $O(mn^{7/4}\beta(n)\log n + n^{11/7+\varepsilon})$ algorithm, which improves an earlier result by Boxer [4]. Our improved bounds can be applied to derive more efficient algorithms for the corresponding variants of this problem (see, e.g., a note to that effect at the end of Section 2).

2. CONGRUENT TRIANGLES IN THREE DIMENSIONS

[1]We follow the convention that an upper bound that involves the parameter ε holds for any $\varepsilon > 0$ and the constant of proportionality depends on ε, and generally tends to infinity as ε tends to 0.

THEOREM 2.1. *Let P be a set of n points in $\mathbb{R}^3$. The number of triangles spanned by P that are congruent to a fixed triangle is $O(n^{5/3} \cdot 2^{\Theta(\alpha^2(n))})$.*

Proof: Let the fixed triangle be $\Delta = x_0y_0z_0$, with side lengths $|x_0y_0| = \xi$, $|x_0z_0| = \eta$, $|y_0z_0| = \zeta$. Let ρ be the distance between z_0 and the line passing through x_0y_0. Fix a pair of points $p, q \in P$ such that $|pq| = \xi$. Let v be a point of P such that pqv is congruent to Δ (with $|pq| = \xi$, $|pv| = \eta$, $|qv| = \zeta$). Let ℓ_{pq} be the line passing through p and q, and let v^* be the projection of v on ℓ_{pq}. Then v^* is independent of v (and depends only on Δ) and any such v lies on a circle γ_{pq} of radius ρ centered at v^* and orthogonal to ℓ_{pq}; see Figure 2. Repeating this analysis for each pair p, q at distance ξ, we obtain a (multi)set $\mathcal{C}$ of congruent circles, one for each such pair of points, and the number of triangles under consideration is equal to the number of incidences between the circles of $\mathcal{C}$ and the points of P. It is easily checked that at most two pairs of points p, q can give rise to the same circle in $\mathcal{C}$, so we may assume that all circles in $\mathcal{C}$ are distinct. Since each circle in $\mathcal{C}$ is generated by a pair of points of P at distance ξ apart, we have, by the results of [6], $|\mathcal{C}| = O(n^{3/2}\beta(n))$, where $\beta(n) = 2^{\Theta(\alpha^2(n))}$ is as above.

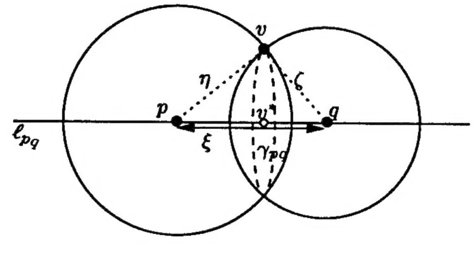

Figure 2: Illustration to the upper bound.

For each $u \in P$, let σ_u denote the sphere of radius η centered at u. Let $\mathcal{S}$ denote the resulting collection of n spheres. Let $P_u = P \cap \sigma_u$ and $\mathcal{C}_u = \{\gamma_{uv} \mid v \in P, |uv| = \xi\}$ (all circles in $\mathcal{C}_u$ lie on σ_u). Put $m_u = |P_u|$ and $c_u = |\mathcal{C}_u|$. We have

$$\begin{aligned} \sum_{u \in P} m_u &= O(n^{3/2}\beta(n)) \qquad (2.1) \\ \sum_{u \in P} c_u &= |\mathcal{C}| = O(n^{3/2}\beta(n)). \end{aligned}$$

We claim that the number of incidences between the points of P_u and the circles of $\mathcal{C}_u$ is

$$O(m_u^{2/3}c_u^{2/3} + m_u + c_u).$$

This follows exactly as in the proof of a similar bound on the number of incidences between points and unit circles in the plane (cf. [6, 16]; in fact, the proof in [16] translates practically verbatim to the case of congruent circles on a sphere).

The number of incidences between the circles of $\mathcal{C}$ and the points of P is thus (using (2.1))

$$O\left[\sum_{u\in P}(m_u^{2/3}c_u^{2/3}+m_u+c_u)\right] =$$
$$O(n^{3/2}\beta(n)) + O\left(\sum_{u\in P} m_u^{2/3}c_u^{2/3}\right).$$

To obtain an upper bound for the second term, we need the following properties.

LEMMA 2.2. *The number of containments between a subset $\mathcal{S}_0$ of spheres of $\mathcal{S}$ and the circles of $\mathcal{C}$ is*

$$O\left(n^{3/4}|\mathcal{S}_0|^{3/4}\beta(n)+n+|\mathcal{S}_0|\right).$$

Proof: Let $P_0 \subseteq P$ denote the set of centers of the spheres of $\mathcal{S}_0$. Consider a containment between a sphere σ_u, for $u \in P_0$, and a circle γ_{uv} of $\mathcal{C}$. Then v is a point of P at distance ξ from u. That is, u lies on the sphere of radius ξ centered at v. Conversely, any such point v gives rise to a circle $\gamma_{uv} \in \mathcal{C}$ that is contained in σ_u. The asserted bound is now an immediate consequence of the bound on the number of incidences between points and unit spheres in $\mathbb{R}^3$, as given in [6]. □

For a given integer $k \geq 0$, let $t_{\geq k} = |P_{\geq k}|$ denote the number of spheres in $\mathcal{S}$ that contain at least k circles of $\mathcal{C}$. An immediate corollary of the previous lemma is the following.

COROLLARY 2.3.

$$t_{\geq k} = |P_{\geq k}| = O\left(\frac{n^3\beta^4(n)}{k^4}+\frac{n}{k}\right). \qquad (2.2)$$

Proof: Let $\mathcal{S}_{\geq k} \subseteq \mathcal{S}$ denote the set of spheres that contain at least k circles of $\mathcal{C}$ ($P_{\geq k}$ is the set of centers of these spheres). The number of sphere-circle containments between the spheres of $\mathcal{S}_{\geq k}$ and the circles of $\mathcal{C}$ is at least $kt_{\geq k}$. Using Lemma 2.2, we have

$$kt_{\geq k} = O\left(n^{3/4}t_{\geq k}^{3/4}\beta(n)+n+t_{\geq k}\right),$$

from which the asserted bound follows easily. □

We now obtain a bound on the expression $\sum_{u\in P} m_u^{2/3}c_u^{2/3}$. Fix a threshold parameter k, whose value will be specified later. We have

$$\begin{aligned}\sum_{u\in P} m_u^{2/3}c_u^{2/3} &= \sum_{u\in P_{<k}} m_u^{2/3}c_u^{2/3} + \sum_{j\geq k}\sum_{u\in P_j} m_u^{2/3}j^{2/3}\\ &\leq k^{2/3}\sum_{u\in P_{<k}} m_u^{2/3} + \sum_{j\geq k} j^{2/3}\sum_{u\in P_j} m_u^{2/3}.\end{aligned}$$

Using Hölder's inequality and (2.1), the first sum is at most

$$\begin{aligned}k^{2/3}\sum_{u\in P_{<k}} m_u^{2/3} &\leq k^{2/3}\left(\sum_{u\in P} m_u\right)^{2/3}\cdot n^{1/3}\\ &= k^{2/3}n^{1/3}\cdot O\left((n^{3/2}\beta(n))^{2/3}\right)\\ &= O(k^{2/3}n^{4/3}\beta^{2/3}(n)).\end{aligned}$$

Using once again Hölder's inequality, in conjunction with (2.1) and (2.2), the second sum can be bounded by

$$\begin{aligned}\sum_{j\geq k} j^{2/3}\sum_{u\in P_j} m_u^{2/3} &\leq \sum_{j\geq k} j^{2/3}\left(\sum_{u\in P_j} m_u\right)^{2/3}|P_j|^{1/3}\\
&\leq \left(\sum_{j\leq k}\sum_{u\in P_j} m_u\right)^{2/3}\cdot\left(\sum_{j\geq k} j^2|P_j|\right)^{1/3}\\
&\leq \left(\sum_{u\in P} m_u\right)^{2/3}\cdot\left(k^2|P_{\geq k}|+\sum_{j>k} j|P_{\geq j}|\right)^{1/3}\\
&= O\left((n^{3/2}\beta(n))^{2/3}\right)\cdot\left[\frac{n^3\beta^4(n)}{k^2}+nk\right.\\
&\qquad\left.+\sum_{j>k}\left(\frac{n^3\beta^4(n)}{j^3}+n\right)\right]^{1/3}\\
&= O(n\beta^{2/3}(n))\cdot\left(\frac{n^3\beta^4(n)}{k^2}+n^2\right)^{1/3}\\
&= O\left(n^{5/3}\beta^{2/3}(n)+\frac{n^2\beta^2(n)}{k^{2/3}}\right).\end{aligned}$$

Hence, the total number of triangles in $f_2^{(3)}(P,\Delta)$ is

$$O\left(k^{2/3}n^{4/3}\beta^{2/3}(n)+n^{5/3}\beta^{2/3}(n)+\frac{n^2\beta^2(n)}{k^{2/3}}\right).$$

Choosing $k = n^{1/2}\beta(n)$, we obtain the asserted bound. □

An immediate corollary of this result is that we can determine, in time $O(mn^{5/3}\beta(n)\log n)$, whether a set S of n points in $\mathbb{R}^3$ has a congruent copy of a set P of m points.

3. CONGRUENT TRIANGLES IN HIGHER DIMENSIONS

We now prove near-optimal bounds on $f_2^{(d)}(n)$, for $d \geq 4$. Recall that the problem is interesting only for $d = 4, 5$ because $f_2^{(d)}(n) = \Theta(n^3)$ for $d \geq 6$. Let P be a set of n points in $\mathbb{R}^d$, and let $\Delta = x_0y_0z_0$ be the fixed triangle, with side lengths $|x_0y_0| = \xi$, $|x_0z_0| = \eta$, and $|y_0z_0| = \zeta$. For a given triple of sets A, B, C of points in $\mathbb{R}^d$, let $\Psi(A,B,C;\Delta)$ denote the set of triangles uvw such that $(u,v,w) \in A\times B\times C$, $|uv| = \xi$, $|uw| = \eta$, and $|vw| = \zeta$. Set $\psi(A,B,C;\Delta) = |\Psi(A,B,C;\Delta)|$ and $\psi^{(d)}(a,b,c) = \max\psi(A,B,C;\Delta)$, where the maximum is taken over all sets A,B,C in $\mathbb{R}^d$ with $|A| = a$, $|B| = b$, and $|C| = c$ and over all triangles Δ. Set $\psi^{(d)}(n) = \psi^{(d)}(n,n,n)$. Obviously, $f_2^{(d)}(P,\Delta) = \psi(P,P,P;\Delta)$ and $f_2^{(d)}(n) \leq \psi^{(d)}(n)$. It therefore suffices to obtain a bound on $\psi^{(d)}(a,b,c)$.

Let A, B, C, and Δ be as defined above. We apply the following randomized divide-and-conquer process, which consists of three substeps. Let r be a sufficiently large constant, depending on ε, whose value will be specified later. In the first step, which we refer to as the *A-step*, we regard A as a set of points but map B and C to spheres. Denote by $\sigma_\rho(x)$ the $(d-1)$-sphere of radius ρ centered at x. With each point $p \in B$ (resp. $q \in C$), we associate the sphere $\sigma_\xi(p)$ (resp. $\sigma_\eta(q)$). Set $\Sigma_B = \{\sigma_\xi(p) \mid p \in B\}$, $\Sigma_C = \{\sigma_\eta(q) \mid q \in C\}$, and $\Sigma = \Sigma_B \cup \Sigma_C$.

A subdivision Ξ of $\mathbb{R}^d$ into constant-description-complexity cells (in the sense defined in [14]) is called a $(1/r)$-cutting of Σ if each cell in Ξ is crossed by at most b/r (resp. c/r) spheres of Σ_B (resp. Σ_C). Using a result of Agarwal *et al.* [1] and the generalized zone theorem by Aronov *et al.* [3], it can be shown that there exists a $(1/r)$-cutting of Σ of size $O(r^d \log r)$. By splitting the cells of Ξ further as necessary, we may assume that each cell contains at most a/r^d points of A.

For each cell $\tau \in \Xi$, let $A_\tau = A \cap \tau$, $B_\tau = \{p \in B \mid \tau \subset \sigma_\xi(p)\}$, and $B^*_\tau = \{p \in B \mid \tau \cap \sigma_\xi(p) \neq \emptyset$ and $\tau \not\subset \sigma_\xi(p)\}$. That is, a point $p \in B$ is in B_τ if the sphere $\sigma_\xi(p)$ contains the (necessarily lower dimensional) cell τ, and it is in B^*_τ if $\sigma_\xi(p)$ crosses τ. Similarly, we define $C_\tau = \{q \in C \mid \tau \subset \sigma_\eta(q)\}$, $C^*_\tau = \{q \in C \mid \tau \cap \sigma_\eta(q) \neq \emptyset \text{ and } \tau \not\subset \sigma_\eta(q)\}$. By construction, $|A_\tau| \leq a/r^d$, $\sum_\tau |A_\tau| = a$, $|B^*_\tau| \leq b/r$ and $|C^*_\tau| \leq c/r$. Since the point sets A, B, and C are not in general position, the subset B_τ (resp. C_τ) could be as large as B (resp. C). Note that B_τ and C_τ can be nonempty only if τ is a lower-dimensional cell.

If a triangle Δuvw is in $\Psi(A, B, C)$, then $u \in \sigma_\xi(v) \cap \sigma_\eta(w)$. If $u \in A_\tau$, then $v \in B_\tau \cup B^*_\tau$ and $w \in C_\tau \cup C^*_\tau$. Therefore,

$$\begin{aligned}
&\psi(A, B, C; \Delta) \\
&\leq \sum_{\tau \in \Xi} \Big[\psi(A_\tau, B^*_\tau, C^*_\tau; \Delta) + \psi(A_\tau, B_\tau, C; \Delta) \\
&\quad + \psi(A_\tau, B, C_\tau; \Delta) \Big] \\
&\leq O(r^d \log r) \cdot \psi^{(d)}\left(\frac{a}{r^d}, \frac{b}{r}, \frac{c}{r}\right) + \\
&\quad \sum_{\tau \in \Xi} \Big[\psi(A_\tau, B_\tau, C; \Delta) + \psi(A_\tau, B, C_\tau; \Delta) \Big].
\end{aligned} \tag{3.1}$$

We now obtain bounds on $\psi(A_\tau, B_\tau, C; \Delta)$ and $\psi(A_\tau, B, C_\tau; \Delta)$ for $d = 4, 5$, and substitute them in the above recurrence to derive the corresponding bounds for the general values of $\psi^{(4)}$ and $\psi^{(5)}$.

3.1 The four-dimensional case

LEMMA 3.1. *Let A, B, and C be three point sets of sizes a, b, c, respectively, in $\mathbb{R}^4$. For any cell τ in the corresponding subdivision Ξ,*

$$\psi(A_\tau, B_\tau, C; \Delta) + \psi(A_\tau, B, C_\tau; \Delta) = O(|A_\tau||B| + |A_\tau||C| + |B||C|).$$

Proof: As noted above, we may assume that τ is a lower dimensional cell.

We first bound $\psi(A_\tau, B_\tau, C; \Delta)$. The assertion is obvious if $\min\{|A_\tau|, |B_\tau|\} \leq 2$, so assume that each of the two sets has at least three points. Recall that each point of A_τ lies at distance ξ from every point of B_τ. This implies that there exist two orthogonal concentric circles γ_A, γ_B such that $A_\tau \subset \gamma_A$ and $B_\tau \subset \gamma_B$; see Figure 3. Indeed, let u_1, u_2, u_3 be three distinct points of A_τ. The intersection of the spheres $\sigma_\xi(u_1)$, $\sigma_\xi(u_2)$, $\sigma_\xi(u_3)$ is a circle; it cannot be a 2-sphere because a 2-sphere can lie on only two 3-spheres of a given radius. Let γ_B denote this intersection circle, and let π be the 2-plane containing γ_B. Clearly, $B_\tau \subset \gamma_B$. The center o of B_τ is such that $u_1 o$, $u_2 o$, $u_3 o$ are all orthogonal

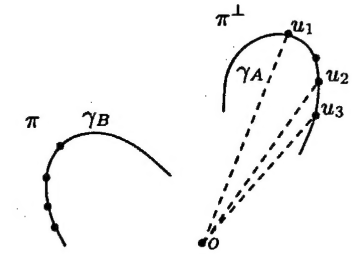

Figure 3: Illustration to the upper bound.

to π. This implies that u_1, u_2, u_3 lie in the (unique) plane $\pi^\perp$ containing o and orthogonal to π. Applying a symmetric argument in which the roles of A_τ and B_τ are reversed completes the proof of the claim.

Let w be any point in C. If w lies at distance η from at most two points of A_τ, then $\psi(A_\tau, B_\tau, \{w\}; \Delta) \leq 2|B_\tau|$, for an overall bound of $2|B_\tau||C|$. Similarly, if w lies at distance ζ from at most two points of B_τ, then $\psi(A_\tau, B_\tau, \{w\}; \Delta) \leq 2|A_\tau|$, for an overall bound of $2|A_\tau||C|$. If w is at distances η and ζ from at least three points of A_τ and B_τ, respectively, then w lies on a circle γ_C that is orthogonal to both γ_A and γ_B. But this is impossible in $\mathbb{R}^4$, so $\psi(A_\tau, B_\tau, C; \Delta) \leq 2(|A_\tau| + |B_\tau|)|C|$. A similar argument shows that $\psi(A_\tau, B, C_\tau; \Delta) \leq 2(|A_\tau| + |C_\tau|)|B|$. Summing all the bounds obtained above, the assertion of the lemma follows. □

In other words, we can write (3.1) for $d = 4$ as

$$\psi(A, B, C; \Delta) = O(r^4 \log r) \cdot \left[(ab + ac + bc) + \psi^{(4)}\left(\frac{a}{r^4}, \frac{b}{r}, \frac{c}{r}\right) \right].$$

We now repeat this analysis a second time, using each of the sets B^*_τ as the set of points and the two other sets as representing sets of spheres of appropriate radii (this is the *B-step*). Then we perform a third step, the *C-step*, in which the resulting susbsets of C represent points and the two other subsets represent spheres. In each of the second and third steps, the size of each set of spheres decreases by a factor of r, and the size of each set of points decreases by a factor of r^4. After the third round, we have $O(r^{12} \log^3 r)$ subproblems in which the size of each point set has been reduced by a factor of r^6. Therefore we obtain the following recurrence:

$$\psi^{(4)}(n) = O(r^{12} \log^3 r) \psi^{(4)}\left(\frac{n}{r^6}\right) + O(n^2), \tag{3.2}$$

where the constant of proportionality of the second term depends (polynomially) on r. For any constant $\varepsilon > 0$, with an appropriate choice of r as a function of the prescribed ε, it can be shown that the solution to (3.2) is $\psi^{(4)}(n) = O(n^{2+\varepsilon})$, where the constant of proportionality depends on ε. Applying this bound for $A = B = C = P$, we obtain that $f_2^{(4)}(n) = O(n^{2+\varepsilon})$. It can be shown that $f_2^{(4)}(n) = \Omega(n^2)$, by generalizing Lenz' construction. In fact, it can be shown that this lower bound can be attained for *any* given triangle Δ. Hence, we have the following theorem.

THEOREM 3.2. *Let P be a set of n points in $\mathbb{R}^4$. The number of triangles spanned by P that are congruent to a fixed triangle is $O(n^{2+\varepsilon})$, for any $\varepsilon > 0$, and can be $\Omega(n^2)$ in the worst case.*

3.2 The five-dimensional case

An argument similar but somewhat more involved than the one used in Lemma 3.1 implies the following lemma for $d = 5$.

LEMMA 3.3. *Let A, B, and C be three point sets of sizes a, b, c, respectively, in $\mathbb{R}^5$. For any cell τ in the corresponding subdivision Ξ,*

$$\begin{aligned}&\psi(A_\tau, B_\tau, C; \Delta) + \psi(A_\tau, B, C_\tau; \Delta)\\ &\quad= O(|A_\tau|(|B|^{2/3}|C|^{2/3} + |B| + |C|) + |B||C|).\end{aligned}$$

Proof: The proof follows the same line as that of Lemma 3.1. We first bound $\psi(A_\tau, B_\tau, C; \Delta)$. Again, we can assume that $|A_\tau|, |B_\tau| \geq 3$. Since each point of A_τ lies at distance ξ from every point of B_τ, it follows, similar to the 4-dimensional case, that only two cases are possible:

(i) A_τ lies on a circle γ_A and B_τ lies on a concentric orthogonal 2-sphere φ_B.

(ii) A_τ lies on a 2-sphere φ_A and B_τ lies on a concentric orthogonal circle γ_B.

Indeed, take three distinct points $u_1, u_2, u_3 \in A_\tau$. Arguing as above, B_τ is contained in a 2-sphere that is concentric with and orthogonal to the circle γ that passes through u_1, u_2, u_3. If B_τ contains at least four noncoplanar points then the entire A_τ must be contained in γ, and we get the situation in case (i). Otherwise, the entire B_τ must lie on a single circle and we get the situation in case (ii).

Let w be any point in C. If w lies at distance η from at most three points of A_τ then $\psi(A_\tau, B_\tau, \{w\}; \Delta) \leq 3|B_\tau|$, for an overall bound of $3|B_\tau||C|$. So assume that w is at distance η from at least four points of A_τ.

In case (i), w must lie on a 2-sphere φ_C that is concentric with and orthogonal to γ_A, and thus lies in the same 3-space containing φ_B. We have thus reduced the problem to the following one: We have two concentric spheres, φ, φ', in three dimensions and two finite point sets Q, Q', with $Q \subset \varphi$ and $Q' \subset \varphi'$, and we wish to bound the number of pairs of points in $Q \times Q'$ that are at distance ζ from each other. We claim that the number of such pairs is $O(|Q|^{2/3}|Q'|^{2/3} + |Q| + |Q'|)$. This is proved exactly as in the analysis in [6] of the number of repeated distances in a planar point set, and as in the proof of Theorem 2.1. In other words, the number of triangles under consideration is

$$O\left(|A_\tau|(|B_\tau|^{2/3}|C|^{2/3} + |B_\tau| + |C|)\right).$$

In case (ii), w must lie on a circle γ_C that is concentric with and orthogonal to φ_A, and thus lies in the same 2-plane containing γ_B. In this case it is easily seen that the number of pairs of points in $B_\tau \times (C \cap \gamma_C)$ at distance ζ from each other is at most $2|B_\tau|$, so the number of triangles under consideration is $O(|A_\tau||B_\tau|)$.

The estimation of $\psi(A_\tau, B, C_\tau; \Delta)$ is fully symmetric, and yields the bound

$$O\left(|A_\tau|(|C_\tau|^{2/3}|B|^{2/3} + |C_\tau| + |B|) + |B_\tau||C|\right).$$

Summing all the bounds obtained above, the assertion of the lemma follows. □

We now apply Lemma 3.3 to each lower-dimensional cell $\tau \in \Xi$, sum up the resulting bounds, and recall that r is a constant, to conclude that the number of triangles that satisfy the assumptions of the lemma, over all cells τ, is $O(a(b^{2/3}c^{2/3} + b + c) + bc)$.

Hence, applying a round-robin decomposition process, as in the 4-dimensional case, we obtain the following recurrence for $\psi^{(5)}(n)$:

$$\psi^{(5)}(n) = O(r^{15}\log^3 r)\psi^{(5)}\left(\frac{n}{r^7}\right) + O(n^{7/3}). \qquad (3.3)$$

Using induction on n and choosing a sufficiently large constant value for r, it can be shown that the solution to (3.3) is $\psi^{(5)}(n) = O(n^{7/3})$. A matching lower bound is constructed as follows. Take a unit 2-sphere σ and a unit circle γ that are concentric and orthogonal. Place $n/2$ points on σ so that there are $\Omega(n^{4/3})$ pairs of these points at distance $\sqrt{2}$ apart (as in [8]), and place $n/2$ points arbitrarily on γ. We obtain a set of n points with $\Omega(n^{7/3})$ equilateral triangles of side length $\sqrt{2}$. We thus obtain the following theorem.

THEOREM 3.4. *Let P be a set of n points in $\mathbb{R}^5$. The number of triangles spanned by P that are congruent to a fixed triangle is $O(n^{7/3})$, and the bound is tight in the worst case.*

Remark 3.5 The number of congruent triangles in a set of n points in the plane is $O(n^{4/3})$, which is an immediate consequence of the same bound for the number of repeated distances in the plane. It is curious to note that each of these four bounds is close to $O(n^{(d+2)/3})$, where d is the dimension. However, while for $d = 4, 5$ these bounds are nearly tight (for $d = 4$) and tight (for $d = 5$), they are conjectured not to be tight for $d = 2, 3$.

4. CONGRUENT TETRAHEDRA IN FOUR DIMENSIONS

We now bound the number of tetrahedra spanned by an n-element point set P in $\mathbb{R}^4$ that are congruent to a given tetrahedron $\Delta = pqrs$. Fix three points $u, v, w \in P$ so that the triangle uvw is congruent to the face pqr of Δ. By Theorem 3.2, the number of such triples is $O(n^{2+\varepsilon})$. Any point $z \in P$ such that $uvwz$ is congruent to Δ must lie on a circle γ_{uvw} that is orthogonal to the 2-plane spanned by u, v, w, whose center lies at a fixed point in this plane, which is the image (under the congruence) of the base point s^* of the height of Δ from s.

Let Γ denote the collection of circles γ_{uvw}. Note that the circle γ_{uvw} is fully determined from the points u, v, w, but that it is possible that two different circles γ_{uvw}, $\gamma_{u'v'w'}$ coincide. In this case, $u'v'w'$ is obtained from uvw by a rotation (and/or reflection) in the plane orthogonal to γ_{uvw} about the center of this circle. In other words, all the points $u \in P$ that induce, with two other points of P, a fixed circle $\gamma = \gamma_{uvw}$ so that u maps to p, must lie on a circle $C_{\gamma,p}$, which is concentric with and orthogonal to γ. The radius of $C_{\gamma,p}$ is the distance between p and s^*. Similarly, the points that induce γ and map to q (resp. r) lie on a circle $C_{\gamma,q}$ (resp. $C_{\gamma,r}$). The three circles $C_{\gamma,p}, C_{\gamma,q}$, and $C_{\gamma,r}$ are concentric and coplanar. It is easily checked that

any of these three circles uniquely determines γ and vice versa. For simplicity of presentation, we only use one of these three coplanar circles, say $C_{\gamma,p}$. For a circle $\gamma \in \Gamma$, there are $O(|P\cap\gamma|\cdot|P\cap C_{\gamma,p}|)$ tetrahedra $uvwz$ spanned by P such that $z \in \gamma$ and u, v, w lie on the respective orthogonal concentric circles $C_{\gamma,p}$, $C_{\gamma,q}$, $C_{\gamma,r}$. Indeed, once the point u has been chosen (from $P \cap C_{\gamma,p}$), the point v that maps to q must lie on $C_{\gamma,q}$ and must be at distance $|pq|$ from u. There are at most two such points. Similarly there are two candidate points for w in $P \cap C_{\gamma,r}$ and any point in $P \cap \gamma$ is a candidate for z.

Fix a threshold parameter k, whose value will be specified later. If a circle $\gamma \in \Gamma$ contains fewer than k points, then the number of tetrahedra under consideration is at most k times the number of triangles uvw that are spanned by P, are congruent to pqr, and induce the circle $\gamma_{uvw} = \gamma$. Summing this bound over all such "low-degree" circles, we obtain the bound $O(n^{2+\varepsilon}k)$.

The problem can thus be reduced to the following. We have a set P of n points and a collection Π of pairs of concentric orthogonal circles, in which no two pairs have a circle in common, and at least one circle in each pair contains at least k points of P. Our goal is to estimate the sum

$$\sum_{(\gamma,\gamma')\in\Pi} |P\cap\gamma|\cdot|P\cap\gamma'| \le \sum_{(\gamma,\gamma')\in\Pi} \max\{|P\cap\gamma|,\ |P\cap\gamma'|\}^2.$$

The problem of estimating the last sum can be restated as follows: We have the point-set P and a collection $\mathcal{C}$ of circles, so that each circle in $\mathcal{C}$ contains at least k points of P, and our goal is to estimate the sum $\sum_{\gamma\in\mathcal{C}} |P\cap\gamma|^2$.

LEMMA 4.1. *The number of incidences between a set P of n points and a set $\mathcal{C}$ of t circles in $\mathbb{R}^4$ is $O(n^{3/5}t^{4/5}+n+t)$.*

Proof: The analysis is similar to the one used in [6] to obtain the same bound for the planar case. First, the point-circle incidence graph does not contain $K_{3,2}$ as a subgraph (with 3 points and 2 circles), so the incidence graph can have at most $O(nt^{2/3}+t)$ edges. We then project P and $\mathcal{C}$ onto some generic 2-plane, and apply the divide-and-conquer analysis of [6] to the projected points and curves, to obtain the asserted bound. A similar proof is also given in [2]. □

LEMMA 4.2. *The number $t_{\ge j}$ of circles in $\mathcal{C}$ that contain at least j points of P is*

$$O\left(\frac{n^3}{j^5}+\frac{n}{j}\right).$$

Proof: The number of incidences between these $t_{\ge j}$ circles and the points of P is at least $jt_{\ge j}$. Using Lemma 4.1, we thus have $jt_{\ge j} = O(n^{3/5}t_{\ge j}^{4/5} + n + t_{\ge j})$, from which the asserted bound follows easily. □

Let t_j denote the number of circles in $\mathcal{C}$ that contain exactly j points of P. We then have

$$\begin{aligned}\sum_{\gamma\in\mathcal{C}} |P\cap\gamma|^2 &= \sum_{j\ge k} j^2 t_j = k^2 t_{\ge k} + \sum_{j>k}(2j+1)t_{\ge j} \\ &= O\left(nk+\frac{n^3}{k^3}+\sum_{j\ge k}\left[\frac{n^3}{j^4}+n\right]\right) \\ &= O\left(n^2+\frac{n^3}{k^3}\right).\end{aligned}$$

Hence, the overall number of tetrahedra spanned by P and congruent to Δ_0 is

$$O\left(n^2+\frac{n^3}{k^3}+n^{2+\varepsilon}k\right).$$

Choosing $k = n^{1/4}$, we obtain the following bound.

THEOREM 4.3. *Let P be a set of n points in $\mathbb{R}^4$. The number of tetrahedra spanned by P that are congruent to a fixed tetrahedron is $O(n^{9/4+\varepsilon})$, for any $\varepsilon > 0$.*

5. THE GENERAL CASE

Let P be a set of n points in $\mathbb{R}^d$ and let $3 \le k \le d-1$. Let $\Delta = a_1a_2\cdots a_{k+1}$ be a fixed k-simplex. We wish to bound the number of k-simplices spanned by the points of P that are congruent to Δ.

We assume that we are given $k+1$ sets of points in $\mathbb{R}^d$, call them $P_1,\dots,P_{k+1}$. Initially, $P_1 = P_2 = \cdots = P_{k+1} = P$. Let $\Psi_k(P_1,\dots,P_{k+1};\Delta)$ denote the set of $(k+1)$-tuples $(p_1,p_2,\dots,p_{k+1}) \in P_1\times P_2\times\cdots\times P_{k+1}$ such that the k-simplex $p_1p_2\cdots p_{k+1}$ is congruent to Δ and $|p_ip_j| = |a_ia_j|$ for $1\le i<j\le k+1$ (i.e., p_i maps to a_i). Set

$$\psi_k(P_1,\dots,P_{k+1};\Delta) = |\Psi_k(P_1,\dots,P_{k+1};\Delta)|$$

and

$$\psi_k(n_1,\dots,n_{k+1}) = \max\psi_k(P_1,\dots,P_{k+1};\Delta),$$

where the maximum is taken over all $(k+1)$-tuples of sets $P_1,\dots,P_{k+1}$ in $\mathbb{R}^d$ with $|P_i| = n_i$, for $i = 1,\dots,k+1$, and over all k-simplices Δ. For brevity, we will use $\psi_k(n)$ to denote $\psi_k(n,\dots,n)$. The following lemma will be crucial for our analysis.

LEMMA 5.1. *Let P and Q be two point sets in $\mathbb{R}^d$, so that $|P|$, $|Q| \ge d+1$, and so that $|pq| = a$ for each $p\in P$, $q\in Q$, for some fixed a. Then there exist two spheres Γ_P, Γ_Q, of respective dimensions δ_P, δ_Q and centers c_P, c_Q, such that*

(i) $P\subset\Gamma_P$ and $Q\subset\Gamma_Q$;

(ii) $1\le\delta_P,\delta_Q\le d-3$ and $\delta_P+\delta_Q\le d-2$; and

(iii) Γ_P is orthogonal to Γ_Q and both are orthogonal to the segment c_Pc_Q. (If $\delta_P+\delta_Q = d-2$ then $c_P = c_Q$.)

Conversely, the existence of such a pair of spheres implies that all distances $|pq|$, for each $p\in P$ and $q\in Q$, are equal.

Proof: P is contained in the intersection $\Gamma = \bigcap_{q\in Q}\sigma_a(q)$, where $\sigma_a(q)$ is the $(d-1)$-sphere of radius a centered at q. This intersection is a sphere of dimension at most $d-3$. Indeed, two of these (congruent) $(d-1)$-spheres intersect in a $(d-2)$-sphere, which cannot be contained in any other $(d-1)$-sphere of the same radius. Let $\Gamma_P \subseteq \Gamma$ be the smallest-dimensional sphere containing P, and let δ_P denote its dimension. A symmetric argument implies that Q is also contained in some (smallest-dimensional) sphere Γ_Q, of dimension δ_Q. Clearly, $1\le\delta_P,\delta_Q\le d-3$. Let c_P, c_Q denote the respective centers of Γ_P, Γ_Q, and let r_P, r_Q denote their respective radii.

Since the affine hull H_P of P is equal, by assumption, to the affine hull of Γ_P, which is a (δ_P+1)-dimensional

space, there exist $\delta_P + 2$ points, $p_1, \ldots, p_{\delta_P+2}$, of P, and real coefficients $\lambda_1, \ldots, \lambda_{\delta_P+2}$, so that

$$\sum_{i=1}^{\delta_P+2} \lambda_i = 1 \quad \text{and} \quad \sum_{i=1}^{\delta_P+2} \lambda_i p_i = c_P ,$$

or, in other words,

$$\sum_{i=1}^{\delta_P+2} \lambda_i (p_i - c_P) = 0 .$$

Similarly, there exist $\delta_Q + 2$ points, $q_1, \ldots, q_{\delta_Q+2}$, of Q, and coefficients $\mu_1, \ldots, \mu_{\delta_Q+2}$, so that

$$\sum_{j=1}^{\delta_Q+2} \mu_j = 1 \quad \text{and} \quad \sum_{j=1}^{\delta_Q+2} \mu_j (q_j - c_Q) = 0 .$$

We have, for each i, j,

$$\begin{aligned} a^2 &= |p_i - q_j|^2 \\ &= |(p_i - c_P) + (c_P - c_Q) + (c_Q - q_j)|^2 \\ &= r_P^2 + r_Q^2 + |c_P c_Q|^2 + 2(p_i - c_P) \cdot (c_P - c_Q) \\ &+ 2(c_Q - q_j) \cdot (c_P - c_Q) + 2(p_i - c_P) \cdot (c_Q - q_j). \end{aligned}$$

Hence

$$a^2 = \sum_{i=1}^{\delta_P+2} \sum_{j=1}^{\delta_Q+2} \lambda_i \mu_j a^2 = r_P^2 + r_Q^2 + |c_P c_Q|^2,$$

which implies that

$$\begin{aligned} D_{ij} &= (p_i - c_P) \cdot (c_P - c_Q) + (c_Q - q_j) \cdot (c_P - c_Q) \\ &\quad + (p_i - c_P) \cdot (c_Q - q_j) = 0, \end{aligned}$$

for each i, j. Then, for any fixed j, we have

$$\sum_i \lambda_i D_{ij} = (c_Q - q_j) \cdot (c_P - c_Q) = 0,$$

implying that the affine hull H_Q of Γ_Q is orthogonal to $c_P c_Q$. By a symmetric reasoning, the same holds for the affine hull H_P of Γ_P. This also implies that

$$(p_i - c_P) \cdot (c_Q - q_j) = 0,$$

for each i, j, so H_P and H_Q are also orthogonal to each other. This implies that $\delta_P + \delta_Q \leq d - 2$, and thus completes the proof of the first part of the lemma. The converse part is trivial. □

By applying the above lemma inductively, we can prove the following.

COROLLARY 5.2. *Let $P_1, P_2, \ldots, P_\ell$ be k sets of points in $\mathbb{R}^d$, each of size at least $d+1$, so that for all pairs $1 \leq i < j \leq \ell$ and for any $p \in P_i$ and $q \in P_j$, $|pq| = |a_i a_j|$. Then there exist ℓ spheres $\Gamma_1, \ldots, \Gamma_\ell$ of respective dimensions $\delta_1, \ldots, \delta_\ell$ and centers $c_1, \ldots, c_\ell$, such that*

(i) $P_i \subset \Gamma_i$ for each $1 \leq i \leq \ell$;

(ii) $1 \leq \delta_i \leq d - 3$ for every i and $\sum_{i=1}^{\ell} \delta_i \leq d - \ell$ (if $\sum_{i=1}^{\ell} \delta_i = d - \ell$ then $c_1 = \cdots = c_\ell$); and

(iii) for $i \neq j$, Γ_i is orthogonal to Γ_j and all spheres are orthogonal to the affine hull of $c_1, \ldots, c_\ell$.

We extend the divide-and-conquer procedure described in Section 3 to bound ψ_k. Initially, each P_i is an arbitrary set of points in $\mathbb{R}^d$, but at each step the procedure will decompose a problem into subproblems in which some "cliques" of the point sets will satisfy the conditions of Corollary 5.2. We therefore define a generalized version of the function ψ_k by introducing a *weighted graph* $G = (V, E, \lambda)$, where $V = \{1, \ldots, k+1\}$. A pair $(i, j) \in E$ if $|pq| = |a_i a_j|$ for every $p \in P_i$ and $q \in P_j$. We associate a weight function $\lambda : \{1, \ldots, k+1\} \mapsto \{1, \ldots, d\}$ with the vertices of G, which we simply write as a sequence $(\lambda_1, \ldots, \lambda_{k+1})$. Here λ_i is the dimension of the smallest sphere that contains P_i. By Corollary 5.2, G satisfies the following property.

(G) If $\{i_1, \ldots, i_\ell\}$ is a clique in G, then

$$\sum_{j=1}^{\ell} \lambda_{i_j} \leq d - \ell.$$

We now define $\psi_k^{(G)}(n_1, \ldots, n_{k+1})$ to be the maximum value of $\psi_k(P_1, \ldots, P_{k+1}; \Delta)$, taken only over sets $P_1, \ldots, P_{k+1}$ that satisfy the following properties:

(ψ.i) $|P_i| \geq d + 1$ for each $i = 1, \ldots, k+1$;

(ψ.ii) If $\lambda_i < d$ then P_i is contained in a λ_i-dimensional sphere Γ_i (if $\lambda_i = d$, then P_i is an arbitrary set of points in $\mathbb{R}^d$); and

(ψ.iii) If $\{i_1, \ldots, i_\ell\}$ is a clique in G, then $\Gamma_{i_1}, \ldots, \Gamma_{i_\ell}$ are orthogonal to each other, and all of them are orthogonal to the affine hull of their centers.

As a special case, the original bound $\psi_k(n_1, \ldots, n_{k+1})$ can be written as $\psi_k^{(G_0)}(n_1, \ldots, n_{k+1})$, where

$$G_0 = (V, \emptyset, (d, d, \ldots, d))$$

is an empty weighted graph, with no constraints on any P_i.

We apply a round-robin decomposition method to bound $\psi_k^{(G)}(n) = \psi_k^{(G)}(n, \ldots, n)$. Let $P_1, \ldots, P_{k+1}$ be sets satisfying (ψ.i)–(ψ.iii), each of size n. The process consists of $k+1$ rounds, which are then repeated recursively. In the jth round, P_j is regarded as a set of points, and each P_i, for $i \neq j$, is regarded as a set of congruent spheres of radius $|a_i a_j|$. Consider the first round, in which we regard P_1 as a set of points, and let V_1 denote the collection of all vertices $j \neq 1$ of G such that $(1, j) \notin E$. If $V_1 = \emptyset$, we skip the first round altogether (see below for details). If G contains an edge of the form $(1, j)$, then $\lambda_1 \leq d - 3$, and P_1 lies on a λ_1-dimensional sphere Γ_1. We set U_1 to be the affine hull of Γ_1. Otherwise, if $\lambda_1 = d$ then we set $\Gamma_1 = U_1 = \mathbb{R}^d$. Regard any point p in some P_j, for $j \in V_1$, as defining a λ_1-dimensional sphere $\sigma_j(p)$, obtained as the intersection of U_1 with the $(d-1)$-sphere centered at p and having radius $|a_1 a_j|$. Set $\Sigma_j = \{\sigma_j(p) \mid p \in P_j\}$ and $\Sigma = \bigcup_{j \in V_1} \Sigma_j$.

As above, a subdivision Ξ of Γ_1 into constant-description cells is called a $(1/r)$-cutting of Σ if each cell of Ξ is crossed by at most $|\Sigma_j|/r$ spheres of Σ_j for every $j \in V_1$. Arguing as in Section 3, we have

LEMMA 5.3. *For any given parameter $r > 0$, there exists a $(1/r)$-cutting of Σ of size $O(r^{\lambda_1} \log r)$.*

We fix a parameter r_1 and compute a $(1/r_1)$-cutting of Σ. By splitting cells further as necessary, we may assume that

each cell contains at most $n/r_1^{\lambda_1}$ points of P_1; the number of cells is still $O(r_1^{\lambda_1}\log r_1)$, with a larger constant of proportionality. Let Ξ denote the resulting set of cells. For each $\tau\in\Xi$, set $P_1^\tau = P_1\cap\tau$. Obviously

$$\psi_k(P_1,\ldots,P_{k+1};\Delta) = \sum_{\tau\in\Xi}\psi_k(P_1^\tau,P_2,\ldots,P_{k+1};\Delta).$$

Let $\Delta' = a_2\cdots a_{k+1}$ be the facet of Δ opposite to a_1. Let $\overline{G}_i$ denote the weighted subgraph of G induced by the vertices $V\backslash\{i\}$. Fix a cell $\tau\in\Xi$. We say that a point $p_i\in P_i$, for any $i>1$, is *light* in τ if either $|P_1^\tau|\le d$ or p_i is at distance $|a_1a_i|$ from at most d points of P_1^τ; otherwise, it is *heavy* in τ. Let L_i^τ (resp. P_i^τ) be the subset of points of P_i that are light (resp. heavy) in τ, for $i=2,\ldots,k+1$. Let $p_2\cdots p_{k+1}$ be a $(k-1)$-simplex in $\Psi_{k-1}(P_2,\ldots,L_i^\tau,\ldots,P_{k+1};\Delta')$. Since p_i is light in τ, $p_2\cdots p_{k+1}$ contributes at most d simplices to $\Psi_k(P_1^\tau,\ldots,L_i^\tau,\ldots,P_{k+1};\Delta)$. Therefore the light points of P_i contribute at most

$$d\psi_{k-1}^{(\overline{G}_i)}(n,\ldots,n)\le d\psi_{k-1}^{(\overline{G}_i)}(n)$$

simplices, which implies that

$$\begin{aligned}&\psi_k(P_1^\tau,P_2,\ldots,P_{k+1};\Delta)\\ &\quad\le d\sum_{i=2}^{k+1}\psi_{k-1}^{(\overline{G}_i)}(n)+\psi_k(P_1^\tau,P_2^\tau,\ldots,P_{k+1}^\tau;\Delta).\end{aligned}$$

For each $i>1$, let $\overline{P_i^\tau}=\{p\in P_i^\tau \mid \tau\subset\sigma_i(p)\}$, and let $Q_i^\tau=\{p\in P_i^\tau \mid \tau\cap\sigma_i(p)\neq\emptyset \text{ and } \tau\not\subset\sigma_i(p)\}$. That is, a point p is in Q_i^τ if $\sigma_i(p)$ *crosses* τ. By definition, if $i\notin V_1$ then $Q_i^\tau=\emptyset$ and $\overline{P_i^\tau}=P_i^\tau$. Since Ξ is a $(1/r_1)$-cutting of Σ, we have $|Q_i^\tau|\le n/r_1$ for each $i\in V_1$. If a simplex $p_1\cdots p_{k+1}\in\Psi_k(P_1^\tau,\ldots,P_{k+1}^\tau;\Delta)$, then $p_1\in\bigcap_{i=2}^{k+1}\sigma_i(p_i)$. Since $p_1\in\tau$, we have that $p_i\in\overline{P_i^\tau}\cup Q_i^\tau$ for $2\le i\le k+1$. Hence, we obtain (in the first term, for simplicity of notation, S_i denotes Q_i^τ for $i\in V_1$ and $\overline{P_i^\tau}$ for $i\notin V_1$):

$$\begin{aligned}&\psi_k(P_1^\tau,\ldots,P_{k+1}^\tau;\Delta)\le\\ &\quad\sum_{\tau\in\Xi}\psi_k(P_1^\tau,S_2,\ldots,S_{k+1};\Delta)+\\ &\quad\sum_{\tau\in\Xi}\sum_{i\in V_1}\psi_k(P_1^\tau,P_2^\tau,\ldots,\overline{P_i^\tau},\ldots P_{k+1}^\tau;\Delta)\\ &\le O(r_1^{\lambda_1}\log r_1)\psi_k^{(G)}\left(\frac{n}{r_1^{\lambda_1}},\underbrace{n,\ldots,n}_{k-|V_1|},\underbrace{n/r_1,\ldots,n/r_1}_{|V_1|}\right)\\ &\quad+\sum_{\tau\in\Xi}\sum_{i\in V_1}\psi_k(P_1^\tau,P_2^\tau,\ldots,\overline{P_i^\tau},\ldots P_{k+1}^\tau;\Delta).\end{aligned}$$

Fix an $i\in V_1$. As before, if $|\overline{P_i^\tau}|\le d$, then

$$\psi_k^{(G)}(P_1^\tau,\ldots,\overline{P_i^\tau},\ldots,P_{k+1}^\tau;\Delta)\le d\psi_{k-1}^{(\overline{G}_i)}(n).$$

If $|\overline{P_i^\tau}|\ge d+1$, apply Lemma 5.1 to P_1^τ and $\overline{P_i^\tau}$ to conclude the existence of two spheres $\Gamma\supset P_1^\tau$, $\Gamma'\supset\overline{P_i^\tau}$ that satisfy the properties of that lemma. We clearly have $\Gamma\subset\Gamma_1$ and $\Gamma'\subset\Gamma_i$, and proper inclusions are possible. Let δ,δ' denote the respective dimensions of Γ,Γ'. Note that for any $j\notin V_1$, Γ and Γ_j continue to satisfy the properties of Lemma 5.1 (as did Γ_1 and Γ_j, except that the dimension of Γ may be smaller than that of Γ_1). The same holds for any edge (i,i') in G incident to i, with Γ' replacing Γ_i. We now replace G by the augmented weighted graph $G_{+(1,i)}$, whose edge set is $E\cup\{(1,i)\}$, and in which λ_1 is replaced by δ, λ_i by δ', and for $1\le j\neq i$, λ_j is set to the smallest integer s such that P_j^τ lies in an s-sphere. This step does not increase the value of any λ_ℓ. We can thus rewrite the above recurrence as:

$$\begin{aligned}&\psi_k^{(G)}(P_1,\ldots,P_{k+1};\Delta)\le\\ &\quad O(r_1^{\lambda_1}\log r_1)\psi_k^{(G)}\left(\frac{n}{r_1^{\lambda_1}},\underbrace{n,\ldots,n}_{k-|V_1|},\underbrace{n/r_1,\ldots,n/r_1}_{|V_1|}\right)\\ &\quad+\sum_{i=1}^{k+1}O(\psi_{k-1}^{(\overline{G}_i)}(n))+\sum_{i\in V_1}O(\psi_k^{(G_{+(1,i)})}(n)).\end{aligned}\tag{5.1}$$

We now repeat this step for each of the remaining k rounds. In the ith round we compute a $(1/r_i)$-cutting of an appropriate set of spheres (where P_j is mapped to a set of spheres of common radius $|a_ia_j|$ if $(i,j)\notin E$), so that the size of the cutting is $O(r_i^{\lambda_i}\log r_i)$. To derive the final resulting recurrence, we argue as follows. Fix an index $i\in\{1,\ldots,k+1\}$. In the ith round, the size of the ith set in the leading recursive term (i.e., the term that involves the same $\psi_k^{(G)}$ function) is reduced by a factor of $r_i^{\lambda_i}$. At the jth round, for any $j\neq i$, there are two cases: (a) If $(i,j)\notin E$, then the size of P_i in the leading recursive term is reduced by r_j. (b) If $(i,j)\in E$, then P_i does not change. Thus the total size of the ith set in the final leading recursive term is at most $(n/r_i^{\lambda_i})\prod_{(j,i)\notin E}\frac{1}{r_j}$. For each $i=1,\ldots,k+1$, put $r_i=r^{x_i}$, for some sufficiently large constant parameter r, and for exponents $x_i\ge 0$ that are required to satisfy the following $k+1$ inequalities:

$$\lambda_ix_i+\sum_{(j,i)\notin E}x_j\ge 1,\quad\text{for } i=1,\ldots,k+1;$$

that is, we want the size of each set in the final leading recursive term to be at most n/r. Let $A=A(G)$ be the symmetric $(k+1)\times(k+1)$ matrix, defined by

$$A_{ij}=\begin{cases}\lambda_i & i=j,\\ 1 & i\neq j,\ (i,j)\notin E,\\ 0 & i\neq j,\ (i,j)\in E.\end{cases}$$

Define $\zeta(G)$ to be the optimum value of the linear program min $\lambda\cdot\mathbf{x}$ subject to $A\mathbf{x}\ge\mathbf{1}$. Let $\mathbf{x}=(x_1,\ldots,x_{d+1})$ be a vector that attains the minimum. Set $r_i=r^{x_i}$, for $i=1,\ldots,k+1$. Then the leading term of the recurrence becomes $O(r^{\zeta(G)}\log^{k+1}r)\psi_k^{(G)}(n/r)$, and the full recurrence becomes

$$\begin{aligned}&\psi_k^{(G)}(n)\le O(r^{\zeta(G)}\log^{k+1}r)\psi_k^{(G)}\left(\frac{n}{r}\right)\\ &\quad+\sum_{i=1}^{k+1}O(\psi_{k-1}^{(\overline{G}_i)}(n))+\sum_{i\neq j,\ (i,j)\notin E}O(\psi_k^{(G_{+(i,j)})}(n)),\end{aligned}$$

where the weighted graphs $G_{+(i,j)}$ are defined in a manner similar to the definition of $G_{+(1,i)}$, given above. Let $\zeta(d,k)$ denote the maximum value of $\zeta(G)$ over all graphs with $k+1$ vertices satisfying property **(G)**. Then the solution to the above recurrence is $\psi_k^{(G)}(n)=O(n^{\zeta(d,k)+\varepsilon})$, for any $\varepsilon>0$. Unfortunately, so far we were unable to derive a sharp explicit bound on $\zeta(d,k)$, but conjecture the following.

CONJECTURE 5.4. *For any $d\ge 4$ and $k\le d-2$, $\zeta(d,k)\le d/2$.*

For $G = G_0 = (V, \emptyset, (d, \dots, d))$, we have

$$\zeta(G) = d(k+1)/(d+k) \leq d/2 \quad (\text{for } k \leq d-2)$$

by choosing $x_i = 1/(d+k)$ for each $i = 1, \dots, k+1$. We believe that $\zeta(G)$ is maximized when $G = G_0$ and $k = d-2$. While deriving (5.1), if δ or δ' becomes 1, then one can argue that $\psi_k(P_1^\tau, \dots, P_{k+1}^\tau; \Delta) \leq d(\psi_{k-1}^{\overline{G}_1}(n) + \psi_{k-1}^{\overline{G}_i}(n))$. Using this observation and a few others, we prove, using case analysis on the possible matrices $A(G)$, that $\zeta(d,k) \leq d/2$ for $d \leq 7$ and $k \leq d-2$.

The technical difficulty in proving a bound on $\zeta(d,k)$ lies in the fact that, as G is augmented, the number of recursive subproblems decreases, but the size of the point sets in each recursive subproblem is larger than what it was in the unconstrained case. In particular, sets connected in G to the current set do not change at all. The tradeoff between these two "trends" is not obvious.

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